CHAPTER 12 NOTES – SURDS, INDICES AND EXPONENTIALS

Exercise 12A – Surds

A **surd** is a real, irrational radical such as $\sqrt{2}$, $\sqrt{3}$ or $\sqrt{5}$.

A **radical** is any number written under the root sign.

For example, $\sqrt{4}$ is considered a radical, but isn't a surd as we can write it as 2.

Important Properties:

- $\sqrt{a}$ is never negative, so $\sqrt{a} \geq 0$
- $\sqrt{a}$ is only real if $a \geq 0$
- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ for $a \geq 0$ and $b \geq 0$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ for $a \geq 0$ and $b \geq 0$

Example: Write as a single surd or rational number:

a. $\sqrt{2} \times \sqrt{3}$

$= \sqrt{2 \times 3}$

$= \sqrt{6}$

b. $\frac{\sqrt{18}}{\sqrt{6}}$

$= \sqrt{18/6}$

$= \sqrt{3}$

c. $\sqrt{13} \times \sqrt{13}$

$= 13$

d. $(2\sqrt{5})^2$

$= 2\sqrt{5} \times 2\sqrt{5}$

$= 4 \times 5$

$= 20$

Example: Write $\sqrt{18}$ in the form $a\sqrt{b}$ where a and b are integers and a is as large as possible.

$= \sqrt{9 \times 2}$

$= \sqrt{9} \times \sqrt{2}$

$= 3\sqrt{2}$

Example: Write $\sqrt{98}$ in the form $a\sqrt{b}$ where a and b are integers and a is as large as possible.

$= \sqrt{49 \times 2}$
$= \sqrt{49} \times \sqrt{2}$
$= 7\sqrt{2}$

***Exercise 12A.1:** page 323*
***Questions:** all*

Example: Simplify

a. $3\sqrt{3} + 5\sqrt{3}$

$= 8\sqrt{3}$

b. $2\sqrt{2} - 5\sqrt{2}$

$= -3\sqrt{2}$

Example: Simplify

a. $\sqrt{5}(6 - \sqrt{5})$

$= 6\sqrt{5} - 5$

b. $(6 + \sqrt{3})(1 + 2\sqrt{3})$

$= 6 + 12\sqrt{3} + \sqrt{3} + (2 \times 3)$
$= 6 + 13\sqrt{3} + 6$
$= 12 + 13\sqrt{3}$

Example: Simplify

a. $(5 - \sqrt{2})^2$

$= 5^2 - 2 \times 5 \times \sqrt{2} + \sqrt{2}^2$
$= 25 - 10\sqrt{2} + 2$
$= 27 - 10\sqrt{2}$

b. $(7 + 2\sqrt{5})(7 - 2\sqrt{5})$

$= 7^2 - 2^2\sqrt{5}^2$
$= 49 - 4 \times 5$
$= 49 - 20$
$= 29$

***Exercise 12A.2:** page 324*
***Questions:** all*

Division by Surds

We don't like to write numbers with surds on the bottom of fractions. So for any fraction of the form $\frac{b}{\sqrt{a}}$ we remove the surd by multiplying the whole fraction by $\frac{\sqrt{a}}{\sqrt{a}}$.

We are allowed to do this, because we are essentially just multiplying by 1, so we're not changing the number, we are just changing how it looks.

Example: Write with an integer denominator:

a. $\frac{6}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{6\sqrt{5}}{5}$

b. $\frac{35}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$

$= \frac{35\sqrt{7}}{7}$

$= 5\sqrt{7}$

c. $\frac{2}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{2\sqrt{3}}{3 \times 3}$

$= \frac{2\sqrt{3}}{9}$

d. $\frac{2}{(\sqrt{3})^3}$

$= \frac{2}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{2\sqrt{3}}{3 \times 3}$

$= \frac{2\sqrt{3}}{9}$

$* \ \sqrt{x}^3 = x\sqrt{x}$

***Exercise 12A.3:** page 325*
***Questions:** 1*

Things get a little more interesting with these questions when there is either an addition or subtraction sign in the denominator. Remember, we are trying to get rid of the surd on the bottom.

General Rule: to change the look a fraction like $\frac{c}{a+\sqrt{b}}$ multiply by $\frac{a-\sqrt{b}}{a-\sqrt{b}}$.

This will remove the surd on the bottom line. Let's do an example to see how it works.

Example: Write $\frac{5}{3-\sqrt{2}}$ with an integer denominator.

$$\frac{5}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{5(3+\sqrt{2})}{3^2 - \sqrt{2}^2}$$ * DOTS

sign change

$$= \frac{15 + 5\sqrt{2}}{9 - 2}$$

$$= \frac{15 + 5\sqrt{2}}{7}$$

Example: Write $\frac{2\sqrt{7}+1}{\sqrt{7}-3}$ with an integer denominator.

$$\frac{2\sqrt{7}+1}{\sqrt{7}-3} \times \frac{\sqrt{7}+3}{\sqrt{7}+3} = \frac{2\times 7 + 6\sqrt{7} + \sqrt{7} + 3}{\sqrt{7}^2 - 3^2}$$ * FOIL * DOTS

$$= \frac{14 + 7\sqrt{7} + 3}{7 - 9}$$

$$= -\frac{17 + 7\sqrt{7}}{2}$$

Exercise 12A.3: *page 326*
Questions: *2*

Exercise 12C – Index Laws

Most of this is revision from last year, so it's in your head – we just need to bring it back to the front so we can remember it all.

Index Laws:

- $a^m \times a^n = a^{m+n}$
- $\dfrac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
- $a^0 = 1, \ a \neq 0$
- $a^{-n} = \dfrac{1}{a^n}$

Example: Simplify the following:

a. $2^3 \times 2^2$

$= 2^{3+2}$

$= 2^5$

b. $\dfrac{a^4}{a^{-1}}$

$= a^{4--1}$

$= a^5$

c. $(b^3)^{-3}$

$= b^{3\times -3}$

$= b^{-9}$

d. 4×2^p

$= 2^2 \times 2^p$

$= 2^{2+p}$

e. $\dfrac{3^x}{9^y}$

$= \dfrac{3^x}{3^{2y}}$

$= 3^{x-2y}$

f. 25^{x-1}

$= 5^{2(x-1)}$

$= 5^{2x-2}$

Example: Write as powers of 2:

a. 16

$= 2 \times 2 \times 2 \times 2$

$= 2^4$

b. $\dfrac{1}{16}$

$= \dfrac{1}{2^4}$

$= 2^{-4}$

c. 4×2^m

$= 2^2 \times 2^m$

$= 2^{2+m}$

d. 1

$= 2^0$

e. $\dfrac{2^m}{8}$

$= \dfrac{2^m}{2^3}$

$= 2^{m-3}$

Example: Remove the brackets of:

a. $(-3a^2)^4$ **b.** $\left(\frac{3c}{b}\right)^4$ **c.** $\left(\frac{2x^3y^2}{xy}\right)^2$

$= (-3)^4 a^{2\times 4}$

$= 81a^8$

$= \frac{3^4c^4}{b^4}$

$= \frac{81c^4}{b^4}$

$= \frac{2^2 x^{3\times 2} y^{2\times 2}}{x^2y^2}$

$= \frac{4x^{6}y^{4}}{x^2y^2}$

$= 4x^4y^2$

Example: Simplify, giving your answers in simplest rational form:

a. 7^0 **b.** 3^{-2} **c.** $3^0 - 3^{-1}$ **d.** $\left(\frac{5}{3}\right)^{-2}$

$= 1$

$= \frac{1}{3^2}$

$= 9$

$= 1 - \frac{1}{3}$

$= \frac{2}{3}$

$= \frac{5^{-2}}{3^{-2}}$

$= \frac{3^2}{5^2}$

$= \frac{9}{25}$

Example: Write the following without brackets or negative indices:

a. $(2ab)^{-1}$ **b.** $2(ab)^{-1}$ **c.** $\left(\frac{2x^{-3}y}{z}\right)^2$

$= \frac{1}{2ab}$

$= \frac{2}{ab}$

$= \frac{2^2 x^{-3\times 2} y^2}{z^2}$

$= \frac{4x^{-6}y^2}{z^2}$

$= \frac{4y^2}{x^6z^2}$

Exercise 12C: *page 330*
Questions: *1 – 12*

Exercise 12E – Rational Indices

We now look at powers that involve fractions.

Rational Index Laws:

- $\sqrt{a} = a^{\frac{1}{2}}$
- $\sqrt[3]{a} = a^{\frac{1}{3}}$
- $\sqrt[n]{a} = a^{\frac{1}{n}}$
- $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

Example: Write as a single power of 2:

a. $\sqrt[3]{2}$

$= 2^{1/3}$

b. $\frac{1}{\sqrt{2}}$

$= \frac{1}{2^{1/2}}$

$= 2^{-1/2}$

c. $\sqrt[5]{4}$

$= (2^2)^{1/5}$

$= 2^{2 \times 1/5}$

$= 2^{2/5}$

d. $\frac{1}{\sqrt{8}}$

$= \frac{1}{(2^3)^{1/2}}$

$= \frac{1}{2^{3/2}}$

$= 2^{-3/2}$

Example: Use your calculator to evaluate, correct to 6 decimal places:

a. $2^{\frac{7}{5}}$

≈ 2.639016

b. $\frac{1}{\sqrt[3]{4}}$

≈ 0.629961

Example: Without using a calculator, write in simplest rational form:

a. $8^{\frac{4}{3}}$

$= (2^3)^{4/3}$

$= 2^{3 \times \frac{4}{3}}$

$= 2^4$

$= 16$

b. $8^{-\frac{2}{3}}$

$= (2^3)^{-2/3}$

$= 2^{3 \times -\frac{2}{3}}$

$= 2^{-2}$

$= \frac{1}{4}$

c. $27^{-\frac{2}{3}}$

$= (3^3)^{-2/3}$

$= 3^{3 \times -2/3}$

$= 3^{-2}$

$= \frac{1}{9}$

Exercise 12E: *page 334*
Questions: *all*

Exercise 12G – Exponential Equations

An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent.

We want to get both sides of the equation with the same index. We do this because if $a^x = a^k$ then $x = k$.

Example: Solve for x:

a. $2^x = 16$

$2^x = 2^4$

$x = 4$

b. $3^{x+2} = \frac{1}{27}$

$3^{x+2} = \frac{1}{3^3}$

$3^{x+2} = 3^{-3}$

$x + 2 = -3$

$x = -5$

c. $4^x = 8$

$2^{2x} = 2^3$

$2x = 3$

$x = \frac{3}{2}$

d. $9^{x-2} = \frac{1}{3}$

$3^{2(x-2)} = 3^{-1}$

$2(x-2) = -1$

$2x - 4 = -1$

$2x = 3$

$x = 3/2$

e. $4^x + 2^x - 20 = 0$

$2^{2x} + 2^x - 20 = 0$

$(2^x)^2 + 2^x - 20 = 0$ } treat like a quadratic

$(2^x - 4)(2^x + 5) = 0$

$2^x = 4$ or $2^x = -5$

$2^x = 2^2$

$x = 2$

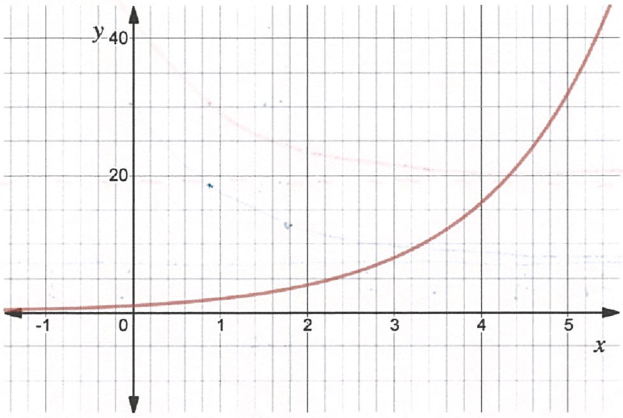
* 2^x cannot be negative *

Exercise 12G: page 338
***Questions:** all*

Exercise 12H – Exponential Functions

We now consider the graphs of **exponential functions**. The 'base' graph which we work from is $y = a^x$ where $a > 0,\ a \neq 1$.

The graph of $y = 2^x$ is below:



As the graphs become more complicated, there are patterns we can use to help us graph them without needing to plot points.

Exponential Function Graphing Rules:

For $y = k \times a^{x-c} + b$

- a controls how steeply the graph increases or decreases
- b controls the vertical translation.
- c controls the horizontal translation

- The equation of the horizontal asymptote is $y = b$

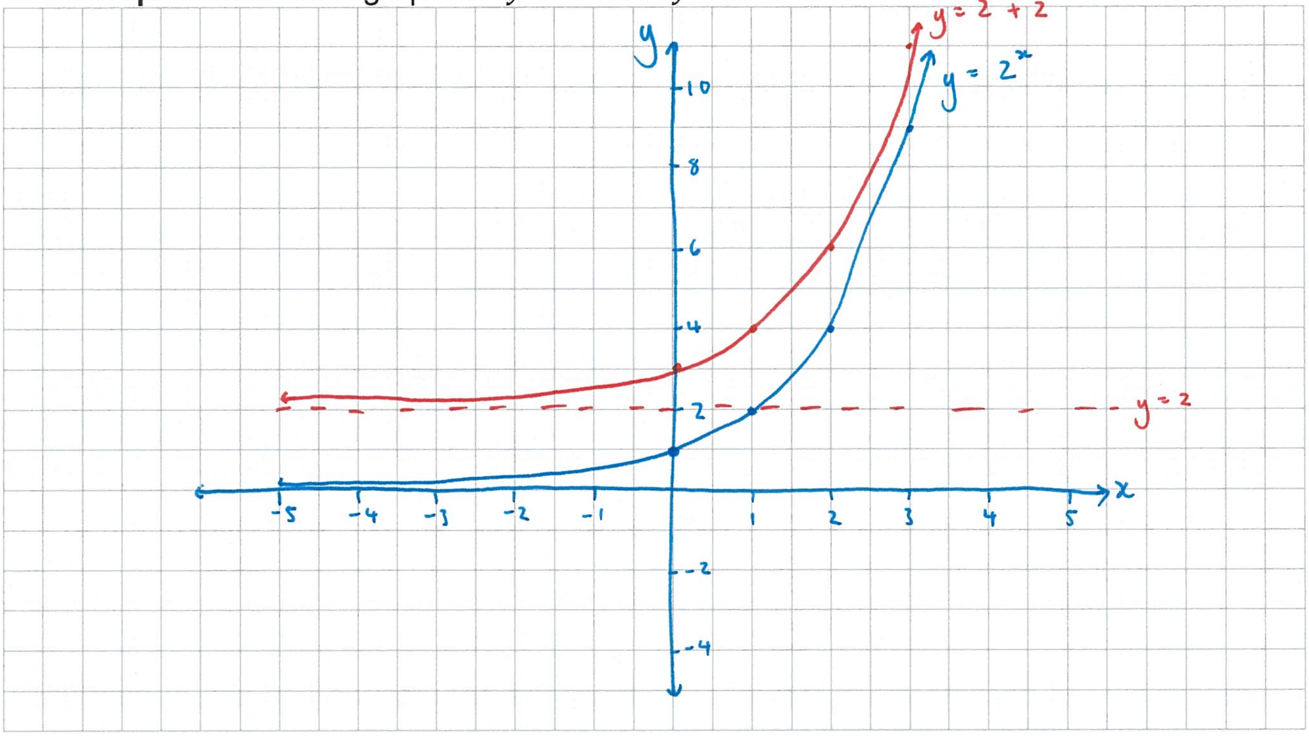
- If $k > 0,\ a > 1$ the function is increasing

- If $k > 0,\ 0 < a < 1$ the function is decreasing

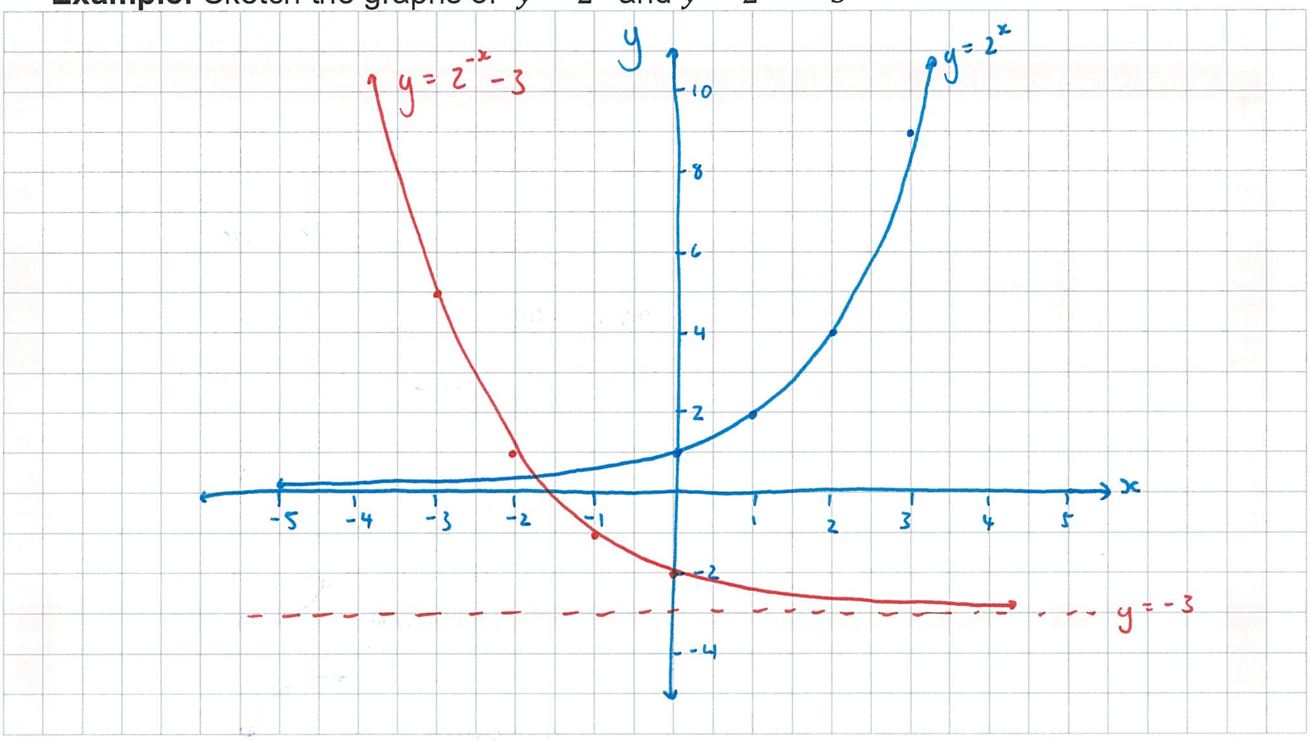
- If $k < 0,\ a > 1$ the function is decreasing

- If $k < 0,\ 0 < a < 1$ the function is increasing

Example: Sketch the graphs of $y = 2^x$ and $y = 2^x + 2$



Example: Sketch the graphs of $y = 2^x$ and $y = 2^{-x} - 3$



Example: Use technology to solve the equation $3^x = 7$

$x = 1.77$

Exercise 12H: *page 342*
Questions: *all*

Exercise 12I – Growth and Decay

We now look at worded problems for this section. Straight to examples!

Growth

Example: An entomologist monitoring a grasshopper plague notices that the area affected by the grasshoppers is given by $A(n) = 1000 \times (1.15)^n$ hectares, where n is the number of weeks after the initial observation.

a. Find the original affected area.

$$A(0) = 1000 \times (1.15)^0$$
$$= 1000 \times 1$$
$$= 1000 \text{ ha}$$

b. Find the affected area after:

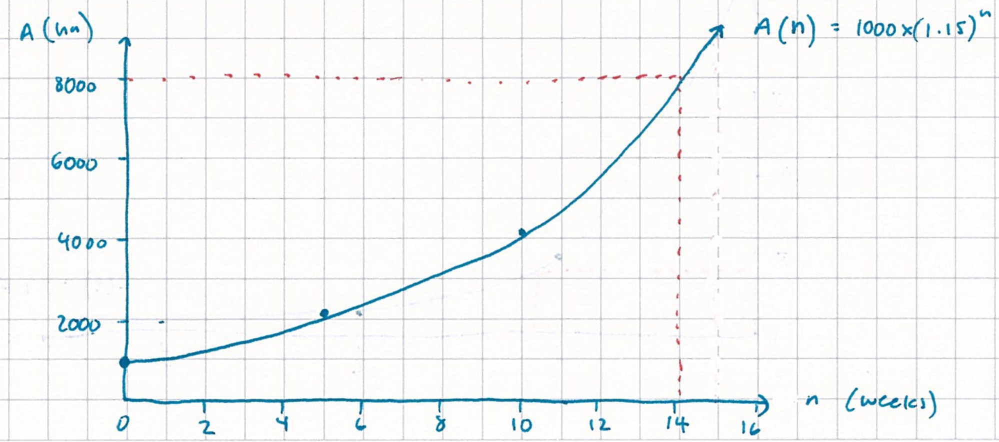
i. 5 weeks

$$A(5) = 1000 \times 1.15^5 \approx 2010 \text{ ha}$$

ii. 10 weeks

$$A(10) = 1000 \times 1.15^{10} \approx 4050 \text{ ha}$$

c. Draw the graph of the affected area over time.



d. Use your graph or technology to find how long it will take for the affected area to reach 8000 hectares.

Using the graph in c) it would take approximately 14 weeks to reach an affected area of 8000 ha

***Exercise 12I.1:** page 345*
***Questions:** all*

Decay

Example: When a diesel-electric generator is switched off, the current dies away according to the formula $I(t) = 24 \times (0.25)^t$ amps, where t is the time in seconds after the power is cut.

a. Find $I(t)$ when $t = 0, 1, 2$ and 3

$I(0) = 24 \times 0.25^0$
$= 24$ amps

$I(2) = 24 \times 0.25^2$
$= 1.5$ amps

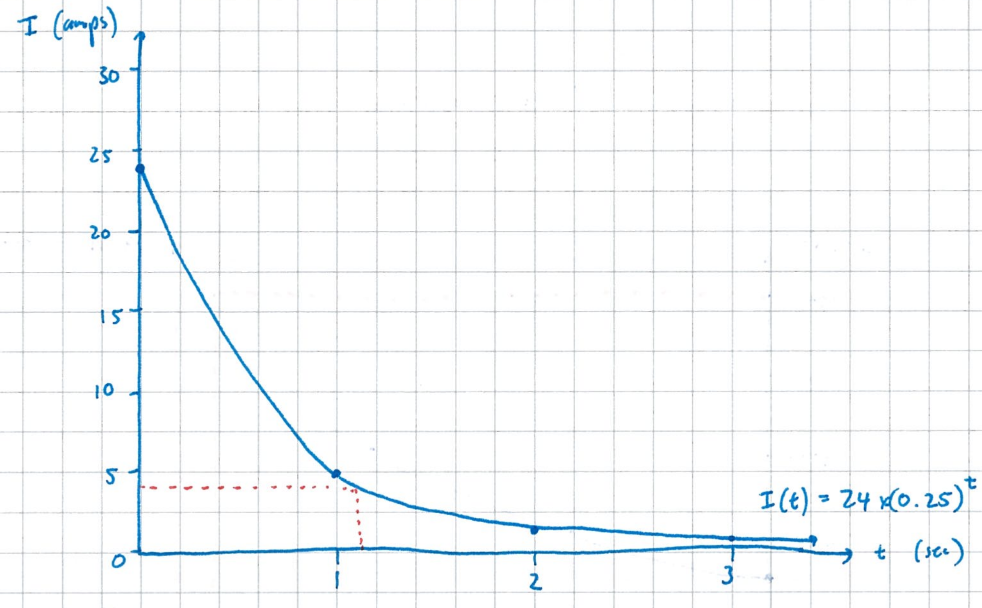
$I(1) = 24 \times 0.25^1$
$= 6$ amps

$I(3) = 24 \times 0.25^3$
$= 0.375$ amps

b. What current flowed in the generator at the instant when it was switched off?

$I(0) = 24$ ie: when generator was off 24 amps flowed through circuit

c. Plot the graph of $I(t)$ for $t \geq 0$ using the information above.



d. Use your graph or technology to find how long it takes for the current to reach 4 amps.

Using graph in part c) it will take approximately 1.1 seconds to reach 4 amps.

***Exercise 12I.2:** page 347*
***Questions:** all*