CHAPTER 14 NOTES – INTRODUCTION TO DIFFERENTIAL CALCULUS

Exercise 14A.1 – Constant Rates of Change

A rate is a comparison between two quantities with different units.

Examples of rates include:

- Speed – it is the rate of change in distance per unit of time.
- A batting average in cricket – 50 runs per wicket.

A constant rate of change means that the increase or decrease per unit of time always remains the same. When we graph this situation, we will get a straight line.

Exercise 14A. 1:
Question 1

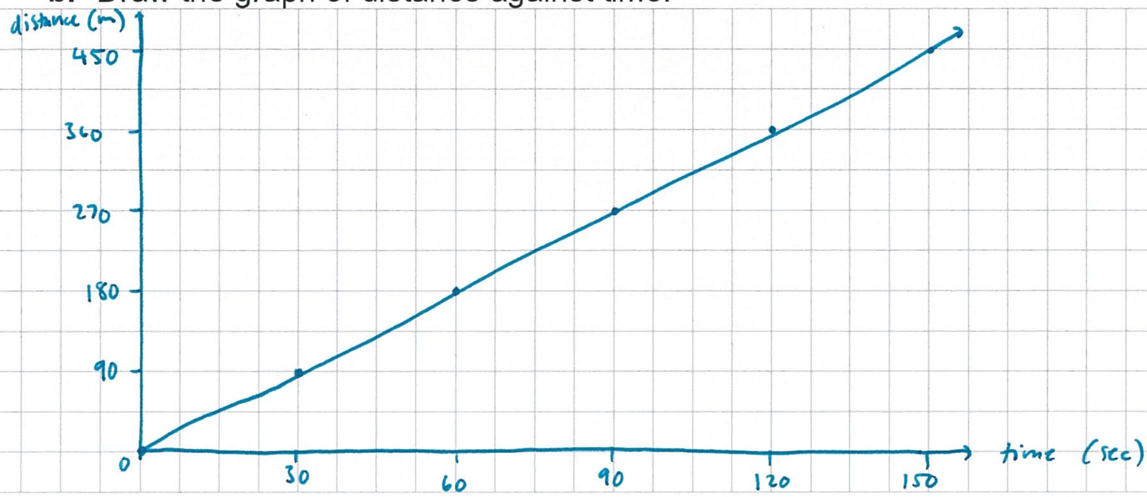
The table below shows the distance travelled by a jogger at 30-second intervals.

Time (sec)	0	30	60	90	120	150
Distance (m)	0	90	180	270	360	450

a. Is the jogger travelling at a constant speed? Explain your answer.

Yes. The distance increases by the same amount each interval.

b. Draw the graph of distance against time.



c. Find the speed of the jogger in metres per second.

Speed $= \frac{90}{30} = 3$ m/s $\quad * \ \frac{y_2 - y_1}{x_2 - x_1} = \frac{90-0}{30-0}$

Exercise 14A.1: *page 375*
Questions: *2 – 4*

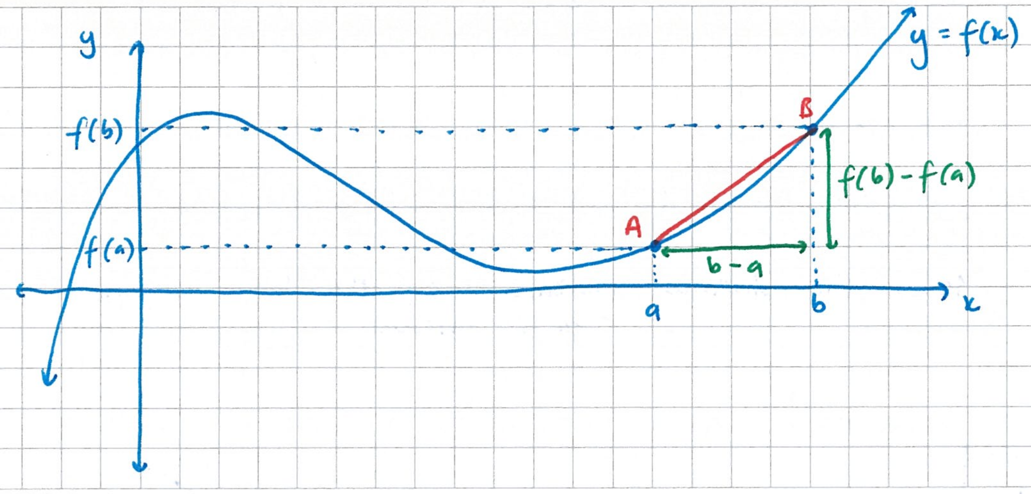
Exercise 14A.2 – Varying Rates of Change

In many real-world situations, rates of change are not constant, but vary over time. For example, no matter how hard I try, my M&M consumption per minute will decrease over time whilst watching a movie.

Even when this is the case, we can find an average rate of change over a period of time. Here comes some notation to make it sound very confusing, but hang in there as the examples will help us understand!

The average rate of change in $f(x)$ from $x = a$ to $x = b$ is $\frac{f(b)-f(a)}{b-a}$

This is the slope of the chord AB in the diagram we are about to draw!



<u>Exercise 14A. 2:</u>
Question 1

Please open your books to page 376 so we can see the diagram we need to refer to.

Aileen is driving from Adelaide to Melbourne. This graph shows the distance travelled against time.

a. Did Aileen travel at constant speed? Explain your answer.

No. The graph is not a straight line. $\therefore$ not constant speed

b. Find Aileen's average speed for:

i. The first 5 hours.

av. speed $= \frac{300 - 0}{5 - 0}$

$= 60$ km/hr

ii. The final 5 hours.

av. speed $= \frac{800 - 300}{10 - 5}$

$= 100$ km/hr

Exercise 14A.2: *page 376*
Questions: *2 – 4*

Exercise 14B – Instantaneous Rates of Change

When we are talking about instantaneous rates of change, we are talking about the change that is happening at a __single__ moment.

A good example is the speedometer in your car. If it reads 65 kmph – well…you are probably speeding, so slow down! But also, that 65 kmph is not your average __speed__, it is the speed you are travelling at that particular __time__.

We can find the instantaneous rate of change in $f(x)$ at any point A by finding the gradient of the __tangent__ at A. It's as easy as that! ☺

Exercise 14B:
Question 1

We will use the graph in the text book again, so please open up to page 380.

The graph shows the distance travelled by a swimmer in a pool. Use the tangents drawn to find the swimmer's instantaneous speed after:

a. 30 seconds

slope of tangent at $x = 30$ is $\frac{1}{2}$

$\therefore$ instantaneous speed is 0.5 m/s after 30 sec.

b. 90 seconds

slope of tangent at $x = 90$ is 2

$\therefore$ instantaneous speed is 2 m/s after 90 sec.

Exercise 14B: *page 380*
Questions: *2, 3*

Exercise 14C – Finding the Gradient of the Tangent

We know how to find the slope of a straight line with equation $y = mx + c$

Formula: $slope = m = \frac{y_2 - y_1}{x_2 - x_1}$

Note that this can be written with slightly different notation.

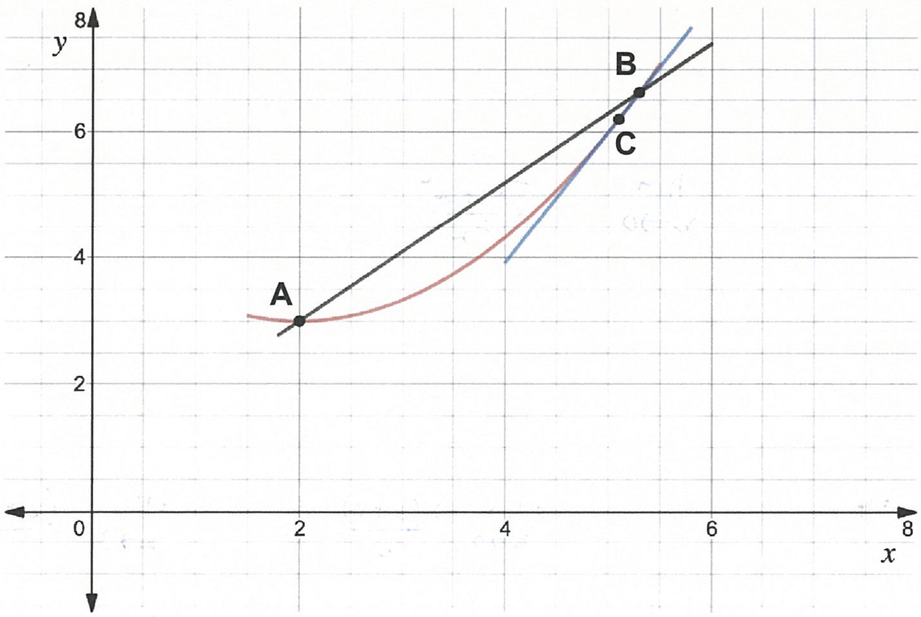
The line with equation $f(x) = mx + c$ has slope formula as follows:

Formula: $slope = m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

But we haven't learnt how to find the slope of a curve.

We know the slope of a curve changes at every point, but we can find the slope at any point we choose using **first principles**. This is also called '**finding the derivative**', and this process also finds the **slope of the** curve at a particular point. The notation for the derivative of a curve $f(x)$ is $f'(x)$.

Consider the following curve. Say we want to find the slope of the curve at point B.



We know we need two points to find the slope, but using the points A and B in the formula $slope = m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ would not be an accurate way to find the slope of the curve at B.

But if A gets closer and closer to B, the formula will give a more accurate slope. Looking at B and C, we can see this is true, as the slope is more accurate. So, the closer the second point gets to B, the more accurate the slope.

But the second point can't be on B as we can't calculate slope with only one point. So, we want the second point as close to B without being on it. We call this a **limit**.

Limits are important, as it is how we can observe the behaviour of the function as it approaches a particular value.

I hear you say "why don't we just work out what is happening at the actual value?" …well, consider the following:

Introduction:

Find $f(0)$ for $f(x) = \frac{5x+x^2}{x}$

The answer will be undefined because of the denominator (the bottom line).

Instead, let's think about what the function will be as x gets really, really close to zero. We first need to factorise the top line so the function looks like:

$$f(x) = \frac{x(5+x)}{x}$$

We then cancel, so we are left with:

$$f(x) = 5 + x$$

Now, think about x being really, really small – the whole thing will equal 5. This is called the **limit**. The notation is looks like this:

$$\lim_{x \to 0} \frac{5x+x^2}{x}$$

Example: Evaluate:

a. $\lim\limits_{x \to 2} x^2$

$= 4$

b. $\lim\limits_{x \to 0} \frac{x^2+3x}{x}$

$= \lim\limits_{x \to 0} \frac{\cancel{x}(x+3)}{\cancel{x}}$

$= \lim\limits_{x \to 0} x+3$

$= 3$

c. $\lim\limits_{x \to 3} \frac{x^2-9}{x-3}$

$= \lim\limits_{x \to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$

$= \lim\limits_{x \to 3} x+3$

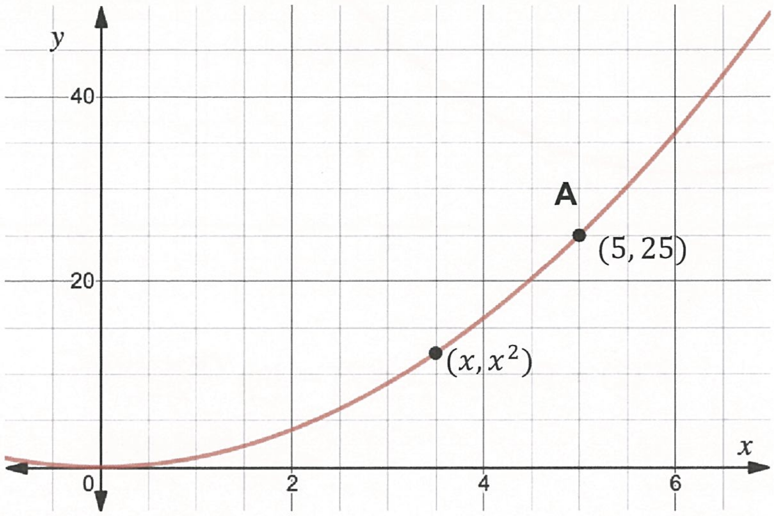
$= 6$

Exercise 14C.1: *page 382*
Questions: *1 – 4*
(every 2nd question)

Exercise 14C.2 – Finding the Gradient of the Tangent at a Particular Point

Let's get straight into looking at an example for this.

Example: Find the slope of the tangent at A on the graph $y = x^2$ using first principles.
(This is the same as finding the slope of the curve at A)



ANSWER:

Point 1: $(5, 25)$ Point 2: (x, x^2)

$$\text{Slope} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{x^2 - 25}{x - 5}$$

$$= \frac{(x+5)(x-5)}{x-5}$$

$$= x + 5$$

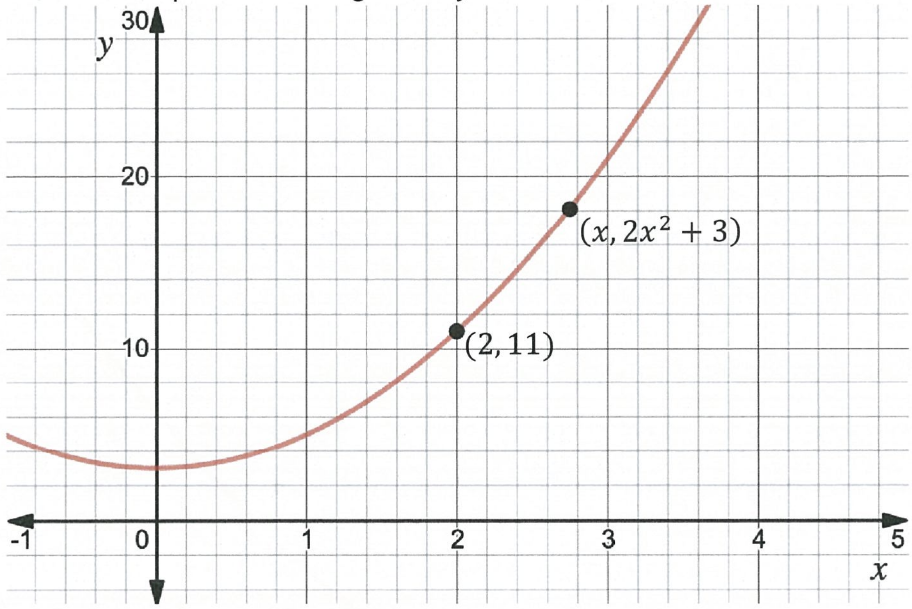
But we know the answer is most accurate when the point (x, x^2) is as close to A as possible. We say that the x coordinate approaches 5. This is written as:

$$\lim_{x \to 5} slope = \lim_{x \to 5} f'(5) = \mathrm{x} + 5$$

$$= 5 + 5$$

$$= 10$$

Example: Find the slope of the tangent to $y = 2x^2 + 3$ when $x = 2$



$$f'(2) = (slope\ at\ x = 2) = \lim_{x \to 2} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \lim_{x \to 2} \frac{(2x^2+3) - 11}{x-2}$$

$$= \lim_{x \to 2} \frac{2x^2 - 8}{x-2}$$

$$= \lim_{x \to 2} \frac{2(x^2-4)}{x-2}$$

$$= \lim_{x \to 2} \frac{2(x+2)(x-2)}{x-2}$$

$$= \lim_{x \to 2} 2(x+2)$$

$$= 2(2+2)$$

$$= 8$$

GENERAL RULE:

Slope at point $\big(a, f(a)\big) = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

This is the **slope of the tangent to the curve** at $x = a$
This is the **slope of the curve** at $x = a$
This is the **derivative of the curve** at $x = a$

Example: Using first principles, find the slope of the tangent to $y = x^2 + 5x$ when $x = 2$

point 1 = $(2, 14)$
point 2 = (x, x^2+5x)

when $x=2$

$$\frac{dy}{dx} = \lim_{x \to 2} \frac{x^2 + 5x - 14}{x - 2}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+7)}{\cancel{x-2}}$$

$$= \lim_{x \to 2} x + 7$$

$$= 9$$

Example: Using first principles, find the slope of the tangent to $y = 3 - x^2$ when $x = -2$

point 1 = $(-2, -1)$
point 2 = $(x, 3 - x^2)$

when $x=-2$

$$\frac{dy}{dx} = \lim_{x \to -2} \frac{3 - x^2 + 1}{x + 2}$$

$$= \lim_{x \to -2} \frac{-x^2 + 4}{x + 2}$$

$$= \lim_{x \to -2} - \frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}}$$

$$= \lim_{x \to -2} -(x - 2)$$

$$= \lim_{x \to -2} -x + 2$$

$$= 4$$

Example: Using first principles, find the slope of the tangent to $y = 4x^2 - x$ when $x = 3$

when $x = 3$

point 1 = $(3, 33)$
point 2 = $(x, 4x^2 - x)$

$$\frac{dy}{dx} = \lim_{x \to 3} \frac{4x^2 - x - 33}{x - 3}$$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(4x+11)}{\cancel{x-3}}$$ (GC) (or splitting)

$$= \lim_{x \to 3} 4x + 11$$

$$= 23$$

Example: Using first principles, find the slope of the tangent to $y = \frac{3}{x}$ when $x = -3$

when $x = -3$

point 1 = $(-3, -1)$
point 2 = $(x, 3/x)$

$$\frac{dy}{dx} = \lim_{x \to -3} \frac{\frac{3}{x} + 1}{x + 3}$$

multiply all by x

$$= \lim_{x \to -3} \frac{3 + x}{x^2 + 3x}$$

$$= \lim_{x \to -3} \frac{\cancel{3 + x}}{x\cancel{(x+3)}}$$

$$= \lim_{x \to -3} \frac{1}{x}$$

$$= \frac{-1}{3}$$

***Exercise 14C.2:** page 384*
***Questions:** 1 – 3*

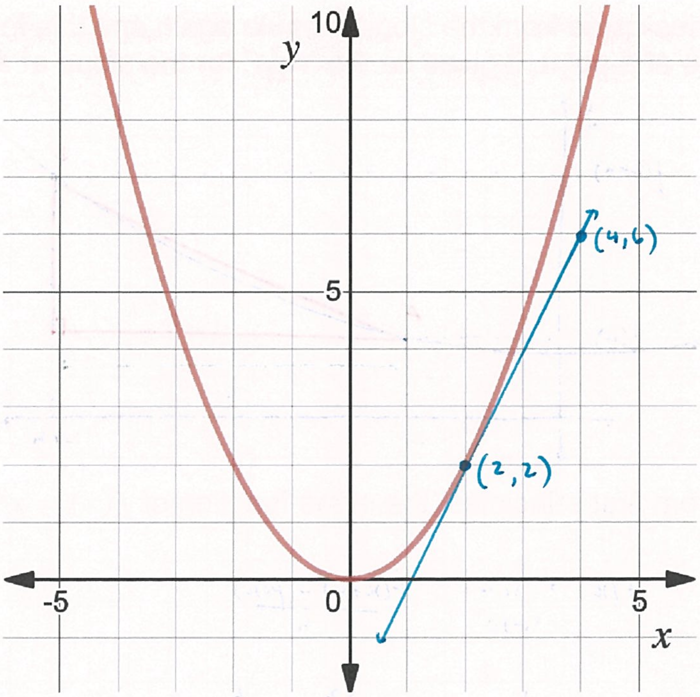
Exercise 14D.1 – The Derivative Function

In the previous exercises, we were using first principals to find the slope at a particular point.

The slope or **gradient function of $f(x)$** is called its ___derivative___ **function**, and is labelled $f'(x)$.

Before we get into using another formula, let's look at these questions below.

Example: For the given graph, find $f'(2)$.



tangent to pt $(2,2)$ passes through $(4,6)$

$\therefore f'(2) = $ gradient of tangent

$= \frac{6-2}{4-2}$

$= \frac{4}{2}$

$= 2$

point 1 $= (2,2)$
point 2 $= (4,6)$

Exercise 14D.1: *page 385*
Questions: *all*

Exercise 14D.2 – Finding the Derivative Function from First Principles

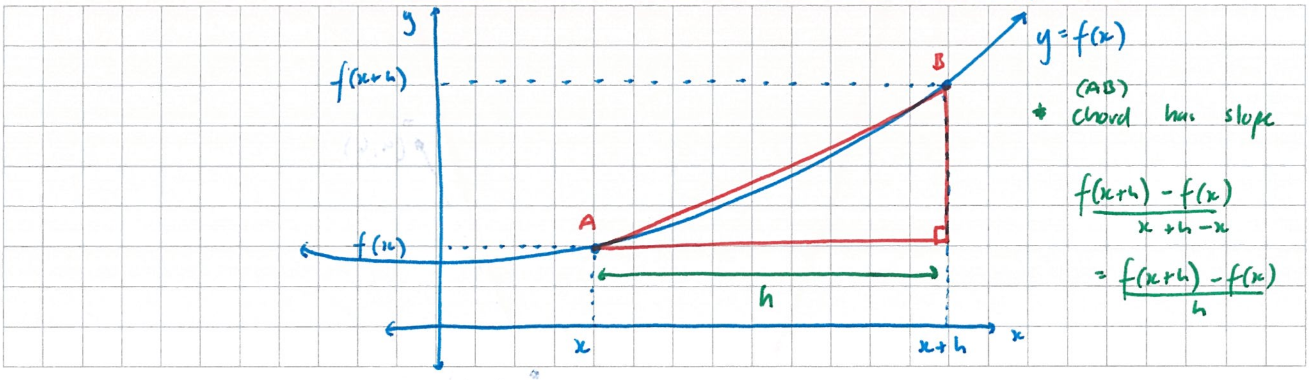
An alternative to the first principles rule we have been working with is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Without going through the whole thing again, this formula is derived in a very similar way to the previous formula. Like earlier, we are finding the slope of a curve, and using two points, however, the two points are now called $(x, f(x))$ and $(x+h, f(x+h))$.

Just like before, we want these points as close to each other as possible, so we want h to be as close to 0 as possible. This is why we have the limit of $h \to 0$.

Note the formula has come from the slope formula again and this formula doesn't give us the actual slope at a point, it gives us the **'rule'** for the slope at a point.



Example: Find, from first principles, the slope function of $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h \qquad (h \neq 0)$$

$$= 2x$$

Example: Find, from first principles, $f'(x)$ if $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \times \frac{x(x+h)}{x(x+h)}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{x - x - h}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}x(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)} \qquad (h \neq 0)$$

$$= \frac{-1}{x^2}$$

Example: Find, from first principles, the slope function of $f(x) = x^4$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h}$$

$$= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4x^3 + 6x^2h + 4xh^2 + h^3)}{\cancel{h}}$$

$$= \lim_{h \to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 \qquad (h \neq 0)$$

$$= 4x^3$$

Example: Find, from first principles, $f'(x)$ if $f(x) = x^2 + 5x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 5(x+h) - x^2 - 5x}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 5x + 5h - x^2 - 5x}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 + 5h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h + 5)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h + 5 \qquad (h \neq 0)$$

$$= 2x + 5$$

We can still find the derivative at an actual point using this method, we just substitute the point in at the very end of our working.

Example: For $f(x) = 3x^2 - 2x$, find $f'(x)$. Hence, find $f'(-2)$ and interpret your answer.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x+h)^2 - 2(x+h) - 3x^2 + 2x}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - 2x - 2h - 3x^2 + 2x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2h - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 2)}{\cancel{h}}$$

$$= \lim_{h \to 0} 6x + 3h - 2 \quad (h \neq 0)$$

$$= 6x - 2$$

$$\therefore f'(-2) = -12 - 2$$
$$= -14$$

$\therefore$ slope of $f(x)$ at $x = -2$ is -14

***Exercise 14D.2:** page 389*
***Questions:** 1 – 10*