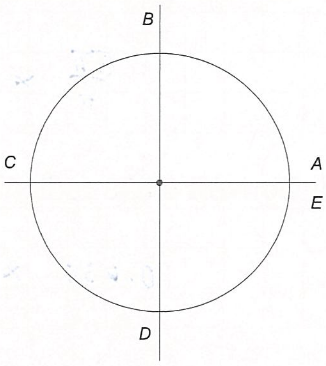
CHAPTER 5 NOTES – THE UNIT CIRCLE AND RADIAN MEASURE

Exercise 5A – Radian Measure

Ever wondered why your calculators are set on 'rad' when you buy them or reset them, and we make you put them back on 'deg'? Well, fear not! Your sleepless nights are over. Today, we talk about radians!

Here we go…

Consider a circle of radius r.



The distance we would walk from A to E if moving anticlockwise would be $2\pi r$

To get from **A to B**, the distance would be $\frac{1}{4}\, 2\pi r$ = $\frac{2}{4}\pi r$ = $\frac{1}{2}\pi r$

To get from **A to C**, the distance would be $\frac{1}{2}\, 2\pi r$ = $\frac{2}{2}\pi r$ = πr

To get from **A to D**, the distance would be $\frac{3}{4}\, 2\pi r$ = $\frac{6}{4}\pi r$ = $\frac{3}{2}\pi r$

So, we can introduce a new measure of angle size called a **radian**.

One radian is the angle swept out by one radius unit around a circle and is approximately equivalent to 57.3°.

Note that 1 radian is the same, regardless of the radius of the circle.

Degree	Radian
90°	$\frac{\pi}{2}$
180°	π
270°	$\frac{3\pi}{2}$
360°	2π

In much of what we do, it is easier to work in radians instead of degrees.

To convert between radians and degrees:

Degrees → Radians	$\times \frac{\pi}{180}$
Radians → Degrees	$\times \frac{180}{\pi}$

Example: Convert $45°$ to radians in terms of π

$$45 \times \frac{\pi}{180} = \frac{\pi}{4}^{c}$$

Example: Convert $126.5°$ to radians

$$126.5 \times \frac{\pi}{180} = \frac{253\pi}{360} = 2.2^{c}$$

Example: Convert $\frac{5\pi}{6}$ radians to degrees

$$\frac{5\pi}{6} \times \frac{180}{\pi} = 150°$$

Example: Convert 0.638 radians to degrees

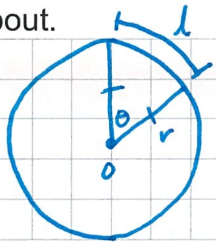
$$0.638 \times \frac{180}{\pi} = 36.55°$$

Exercise 5A: page 122
Questions: all

Exercise 5B – Arc Length and Sector Area

The arc ___length___ is the distance around the circular part of the sector. The formula that we are used to, $l = \frac{\theta}{360} \times 2\pi r$, makes sense because this sector is part of a ___circle___.

Let's draw what we are talking about.



$$l = \frac{\theta}{360} \times 2\pi r$$

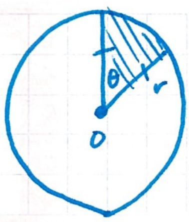
OR $l = \theta r$

When we are working in radians, the formula is:

$$Arc\ length = \theta r$$

To find the area of a sector, we are finding the area of part of a ___circle___. This is why the formula we are used to, $A = \frac{\theta}{360}\pi r^2$, is based on the area of a circle.

Another drawing:



$$A = \frac{\theta}{360}\pi r^2$$

OR $A = \frac{1}{2}\theta r^2$

When we are <u>working in radians</u>, the formula is:

$$Area\ of\ sector = \frac{1}{2}\theta r^2$$

Example: A sector has radius 12 cm and angle 3 radians. Find its:

a. arc length

$l = \theta r$
$= 3 \times 12$
$= 36\text{cm}$

b. area

$A = \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 3 \times 12^2$
$= 216\ \text{cm}^2$

Example: A sector has radius 8.2 cm and arc length 12.3 cm. Find its:

a. angle in radians

$l = \theta r$
$12.3 = \theta \times 8.2$
$\theta = \frac{12.3}{8.2}$
$\theta = 1.5^c$

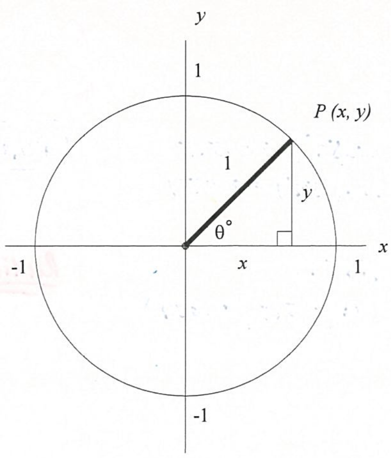
b. area

$A = \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 1.5 \times 8.2^2$
$= 50.43\ \text{cm}^2$

***Exercise 5B:** page 124*
***Questions:** all*

Exercise 5C – The Unit Circle and the Trigonometric Ratios

The unit circle is the circle with centre $(0, 0)$ and radius 1 unit.



If $P(x, y)$ moves around the unit circle such that OP makes an angle of θ with the positive x-axis then:

$$\cos\theta = \frac{adj}{hyp}$$

$$\cos\theta = \frac{x}{1}$$

Also $\sin\theta = \frac{opp}{hyp}$

$$\sin\theta = \frac{y}{1}$$

So the x-coordinate of P is x

So the y-coordinate of P is y

Notice that old man **Pythagoras** comes to the party with $x^2 + y^2 = 1$

Replacing the x and y, we get $\cos^2\theta + \sin^2\theta = 1$

Also notice that $-1 \le x \le 1$ and $-1 \le$ <u>y</u> ≤ 1

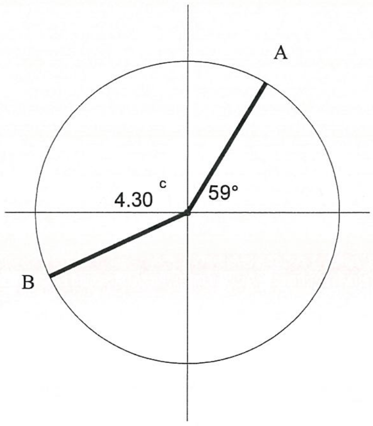
Therefore $-1 \le \cos\theta \le 1$ and $-1 \le$ <u>$\sin\theta$</u> ≤ 1

Hopefully this last statement makes sense to you from the trig you have done before. Anytime we try to do something on the calculator with sin or cos, it will only work if the fraction is between -1 and 1.

We think about **anticlockwise rotations** being <u>positive</u> and **clockwise rotations** being <u>negative</u>.

Also, as $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we can use the unit circle to find the exact value(s) of $\tan\theta$.

Example: For the angle illustrated, write down the actual coordinate of point A and B. Give your answer in exact form and then to three significant figures.



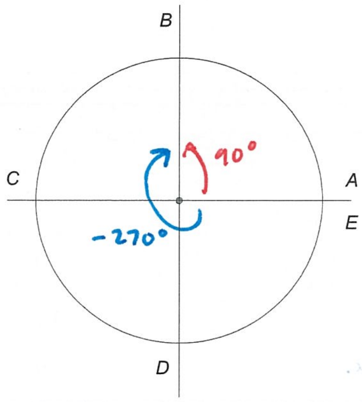
POINT A
x-coord : $\cos(59°) = 0.515$
y-coord $\sin(59°) = 0.857$

Degrees $A(\cos 59, \sin 59)$
or $A(0.515, 0.857)$

POINT B
x-coord : $\cos(4.3^c) = -0.401$
y-coord : $\sin(4.3^c) = -0.916$

Radians $B(\cos 4.3, \sin 4.3)$
or $B(-0.401, -0.916)$

Example: Use a unit circle diagram to find the values of $\cos(-270°)$ and $\sin(-270°)$

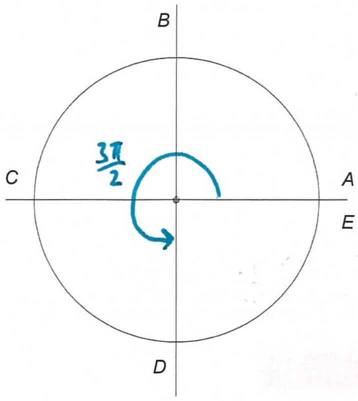


Degrees

x-coord:
$\cos(-270°) = 0$

y-coord:
$\sin(-270°) = 1$

Example: Use a unit circle diagram to find the values of $\cos\left(\frac{3\pi}{2}\right)$ and $\sin\left(\frac{3\pi}{2}\right)$

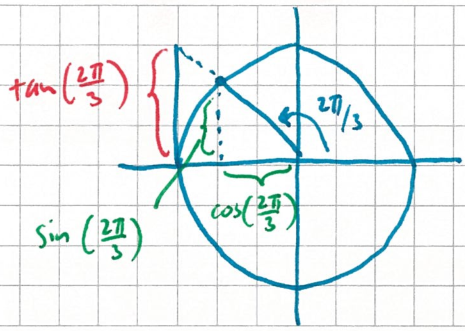


Radians

x-coord: $\cos\left(\frac{3\pi}{2}\right) = 0$

y-coord: $\sin\left(\frac{3\pi}{2}\right) = -1$

Example: Use a unit circle diagram to find the exact value of $\tan\left(\frac{2\pi}{3}\right)$



$$\tan\left(\frac{2\pi}{3}\right) = \frac{\sin(2\pi/3)}{\cos(2\pi/3)} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$$

Now, if we think about the unit circle, there are ___360___ degrees in total for one revolution. If we go around to do a 2nd, 3rd, 4th etc revolution, we are just hitting the same spots again and again. Because of this, we can say that:

$\cos\theta = \cos(\theta + 360k)$ where k is an integer

We typically talking in **radians** though, so the more useful statement is:

$\cos\theta = \cos(\theta + 2k\pi)$ and $\sin\theta = \sin(\theta + 2k\pi)$

The tan function behaves a little differently. Instead of repeating itself every 2π, it actually repeats itself every π. To explain why this is the case, we will look at a couple of specific examples.

Example: For $\theta = \frac{\pi}{6}$, find $\cos\theta$, $\sin\theta$ and $\tan\theta$

$\cos(\pi/6) = \sqrt{3}/2$

$\sin(\pi/6) = 1/2$

$\tan(\pi/6) = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$

Example: Now for an angle which is π more than the angle above, let's find $\cos\theta$, $\sin\theta$ and $\tan\theta$ for $\theta = \frac{7\pi}{6}$. What do you notice?

$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$\sin\left(\frac{7\pi}{6}\right) = \frac{-1}{2}$

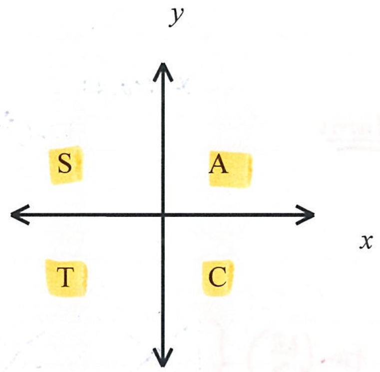
$\tan\left(\frac{7\pi}{6}\right) = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}}$

Sin and cos swap signs but use the same values as the previous example

Summary of Trigonometric Signs:

This diagram tells us where each ratio is positive.

- ALL are positive in Q1
- Sin is positive in Q2
- Tan is positive in Q3
- Cos is positive in Q4



I always remember this by **All Silly Turtles Crawl**, but make up something that you'll remember!

Example:

Quadrant	Degree measure	Radian measure	$\sin\theta$	$\cos\theta$	$\tan\theta$
1	$0 < \theta < 90$	$0 < \theta < \frac{\pi}{2}$	+	+	+
2	$90 < \theta < 180$	$\pi/2 < \theta < \pi$	+	−	−
3	$180 < \theta < 270$	$\pi < \theta < 3\pi/2$	−	−	+
4	$270 < \theta < 360$	$3\pi/2 < \theta < 2\pi$	−	+	−

a. Complete the table.

b. In which quadrants are the following true?

i. $\sin\theta$ is positive

Q.1, Q.2

ii. $\tan\theta$ is negative

Q.2, Q.4

iii. $\cos\theta$ and $\sin\theta$ are both negative

Q3

***Exercise 5C:** page 130*
***Questions:** all*

Exercise 5D.1 – Applications of the Unit Circle

When completing this section, you must keep in mind that $\cos^2\theta + \sin^2\theta = 1$

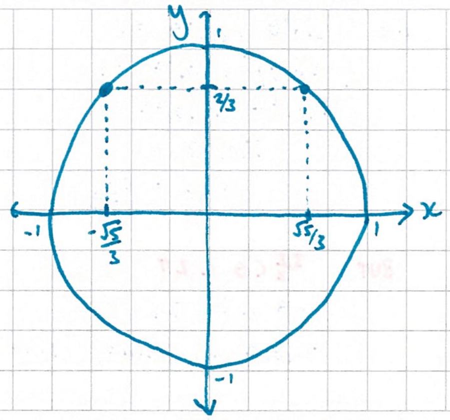
Example: Find the possible values of $\cos\theta$ for $\sin\theta = \frac{2}{3}$ and illustrate your answer on a unit circle.

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta + \left(\tfrac{2}{3}\right)^2 = 1$$

$$\cos^2\theta = \frac{5}{9}$$

$$\cos\theta = \frac{\pm\sqrt{5}}{3}$$



Example: If $\sin\theta = -\frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\cos\theta$ and $\tan\theta$. Give your answers in exact values.

$$\cos^2\theta + \sin^2\theta = 1$$

$$\therefore \cos^2\theta + \left(-3/4\right)^2 = 1$$

$$\cos^2\theta = 1 - 9/16$$

$$\cos^2\theta = 7/16$$

$$\cos\theta = \frac{\pm\sqrt{7}}{4}$$

BUT $\pi < \theta < 3\pi/2$

$\therefore \theta$ is in Q3 (T)

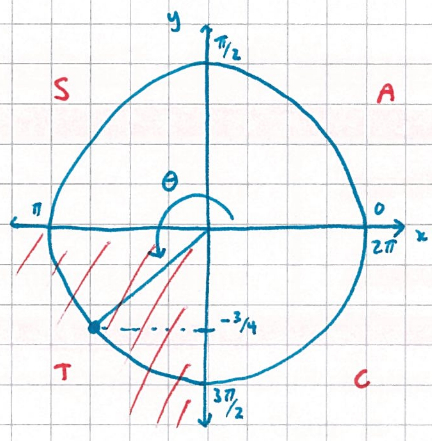
$\therefore \cos\theta$ is negative

$$\therefore \cos\theta = \frac{-\sqrt{7}}{4}$$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{-3/4}{-\sqrt{7}/4}$$

$$= \frac{3}{\sqrt{7}}$$



Example: If $\tan\theta = -2$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\sin\theta$ and $\cos\theta$. Give your answers in exact values.

$\tan\theta = \frac{\sin\theta}{\cos\theta} = -2$

$\therefore \sin\theta = -2\cos\theta$

also $\sin^2\theta + \cos^2\theta = 1$

$(-2\cos\theta)^2 + \cos^2\theta = 1$

$4\cos^2\theta + \cos^2\theta = 1$

$5\cos^2\theta = 1$

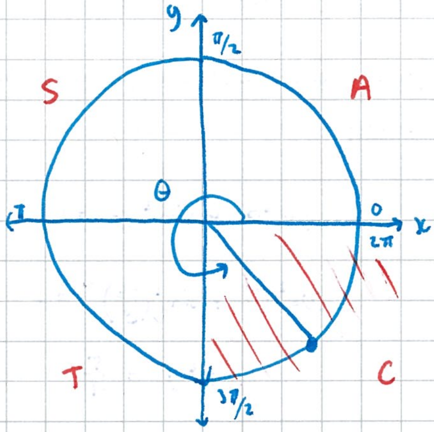
$\cos\theta = \frac{\pm 1}{\sqrt{5}}$

BUT $\frac{3\pi}{2} < \theta < 2\pi$

$\therefore \theta$ is in Q4 (C)

$\therefore \cos\theta$ is positive, $\sin\theta$ is negative

$\therefore \cos\theta = \frac{1}{\sqrt{5}}$ $\quad \sin\theta = \frac{-2}{\sqrt{5}}$



***Exercise 5D.1:** page 134*
***Questions:** all*

Exercise 5D.2 – Finding Angles with Particular Trig Ratios

Degrees	$\sin(180-\theta) = \sin\theta$	$\cos(180-\theta) = -\cos\theta$	$\cos(360-\theta) = \cos\theta$
Radians	$\sin(\pi-\theta) = \sin\theta$	$\cos(\pi-\theta) = -\cos\theta$	$\cos(2\pi-\theta) = \cos\theta$

Example: Find the two angles on the unit circle with $0° \leq \theta \leq 360°$ such that:

a. $\cos\theta = \frac{1}{3}$

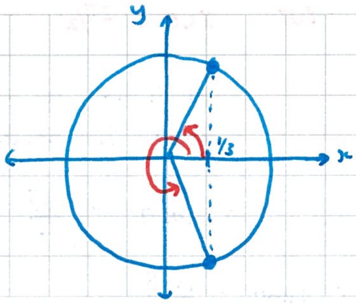
$\theta = \cos^{-1}\left(\frac{1}{3}\right)$

$\theta = 70.53°$

$\therefore \theta = 70.5°$ or $360° - 70.5°$

$\theta = 70.5°$ or $289.5°$

DEGREES



b. $\sin\theta = \frac{3}{4}$

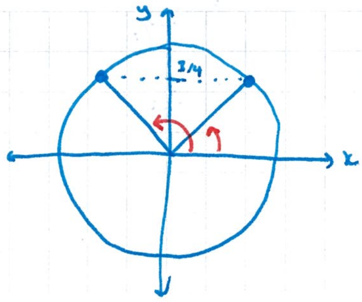
$\theta = \sin^{-1}\left(\frac{3}{4}\right)$

$\theta = 48.59°$

$\therefore \theta = 48.6°$ or $180° - 48.59°$

$\theta = 48.6°$ or $131.4°$

DEGREES



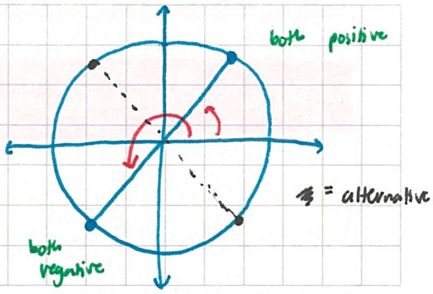
c. $\tan\theta = 2$

$\theta = \tan^{-1}(2)$

$\theta \approx 63.43°$

$\therefore \theta \approx 63.4°$ or $180° + 63.43°$

$\theta \approx 63.4°$ or $243.4°$



DEGREES

Example: Find two angles on the unit circle, with $0 < \theta < 2\pi$ such that:

a. $\sin\theta = -0.4$

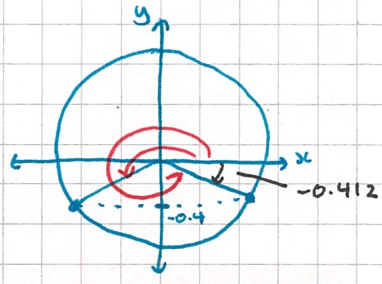
$\theta = \sin^{-1}(-0.4)$

$\theta \approx -0.412$

but $0 \leq \theta \leq 2\pi$

$\therefore \theta \approx \pi + 0.412$ or $2\pi - 0.412$

$\theta \approx 3.55^c$ or 5.87^c



b. $\cos\theta = -\frac{2}{3}$

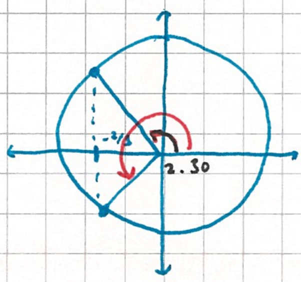
$\theta = \cos^{-1}\left(-\frac{2}{3}\right)$

$\theta \approx 2.30$

but $0 \leq \theta \leq 2\pi$

$\therefore \theta \approx 2.30$ or $2\pi - 2.30$

$\theta \approx 2.30^c$ or 3.98^c



c. $\tan\theta = -\frac{1}{3}$

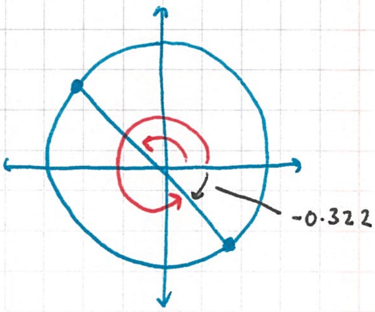
$\theta = \tan^{-1}\left(-\frac{1}{3}\right)$

$\theta \approx -0.322$

but $0 \leq \theta \leq 2\pi$

$\theta \approx \pi - 0.322$ or $2\pi - 0.322$

$\theta \approx 2.82$ or 5.96

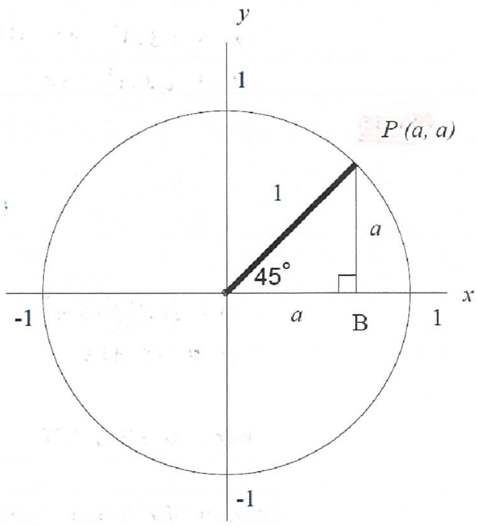


***Exercise 5D.2:** page 135*
***Questions:** all*

Exercise 5E – Multiples of $\pi/6$ and $\pi/4$

Multiples of 45° – $\frac{\pi}{4}$

As the triangle is right angled and a known angle is 45°, the triangle is isosceles.



Let's think about each side being of length a.
Then, to find a, Pythagoras helps us out again:

$a^2 + a^2 = 1^2$

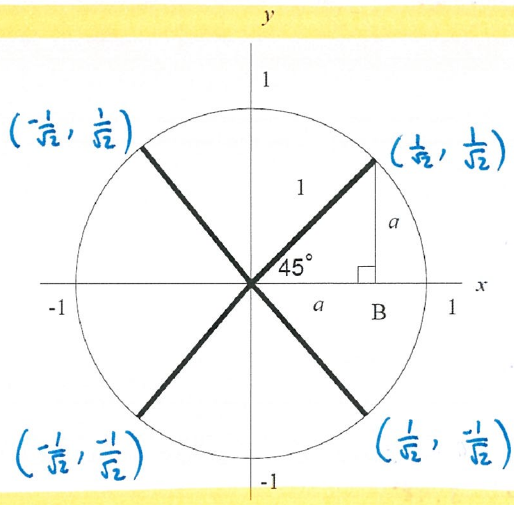
$2a^2 = 1$

$a^2 = \frac{1}{2}$

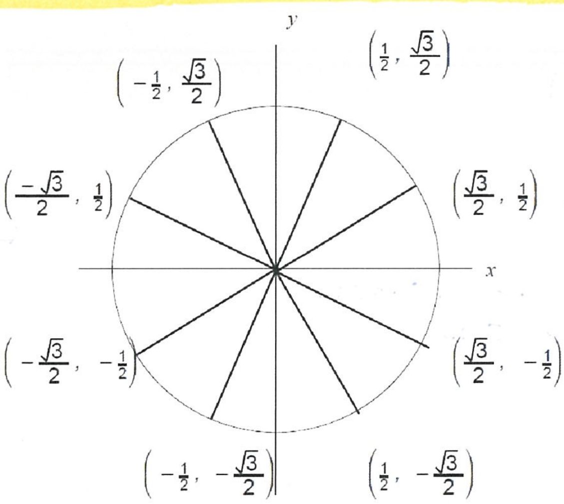
$a = \frac{1}{\sqrt{2}}$ as $a > 0$

So P is the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Summary:



Multiples of 30° and 60° – $\frac{\pi}{6}$ and $\frac{\pi}{3}$



Example:

a. Use a unit circle to find the exact values of $\sin\alpha$ and $\cos\alpha$ for $\alpha = \frac{3\pi}{4}$

$\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$ $\quad\quad$ $\cos\left(\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}}$

b. Now find $\tan\alpha$

$$\tan\left(\frac{3\pi}{4}\right) = \frac{\sin\left(\frac{3\pi}{4}\right)}{\cos(3\pi/4)} = \frac{1/\sqrt{2}}{-1/\sqrt{2}} = -1$$

Example:

a. Use a unit circle diagram to find the exact values of $\cos A$ and $\sin A$ for $\frac{4\pi}{3}$

$\cos\left(\frac{4\pi}{3}\right) = \frac{-1}{2}$ $\quad\quad$ $\sin\left(\frac{4\pi}{3}\right) = \frac{-\sqrt{3}}{2}$

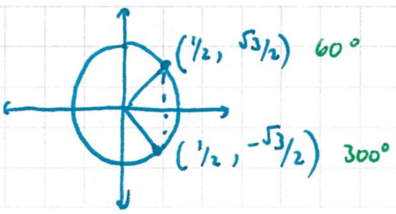
b. Now find $\tan A$

$$\tan\left(\frac{4\pi}{3}\right) = \frac{\sin(4\pi/3)}{\cos(4\pi/3)} = \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3}$$

Example: Without using a calculator, find the value of $8\sin(60^\circ)\cos(150^\circ)$

$$8\sin(60^\circ)\cos(150^\circ) = 8 \times \sqrt{3}/2 \times -\sqrt{3}/2 \quad \text{(unit circle)}$$
$$= 8 \times \frac{-3}{4}$$
$$= \frac{-24}{4}$$
$$= -6$$

Example: Use a unit circle diagram to find all angles between 0˚ and 360˚ which have a cosine of $\frac{1}{2}$

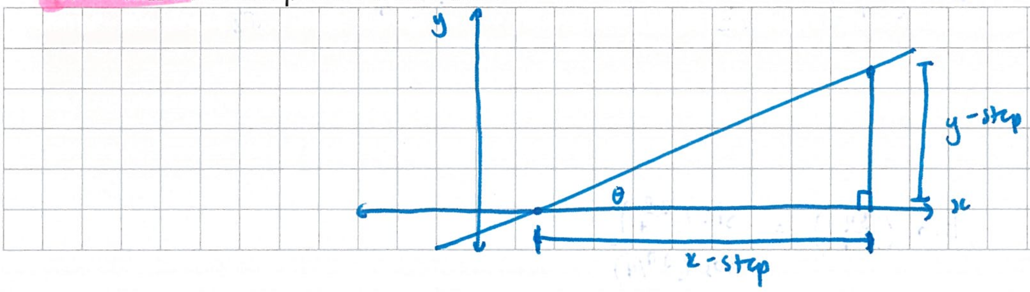


60° and 300° have cosine of ½

***Exercise 5E:** page 139*
***Questions:** all*

Exercise 5F – The Gradient of a Straight Line

If a straight line is inclined at an angle $\underline{\theta}$ to the positive x-axis, then its gradient is $m = \tan\theta$. Let's prove this below:



$$\text{gradient} = \frac{y\text{-step}}{x\text{-step}}$$

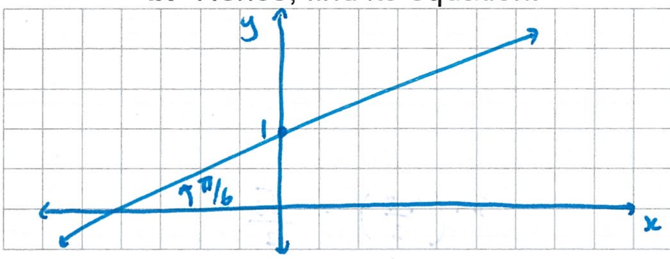
$$m = \frac{\sin\theta}{\cos\theta}$$

$$m = \tan\theta$$

Example:

a. Find the gradient of the line we are about to draw.

b. Hence, find its equation.



a) $m = \tan \pi/6$

$= 1/\sqrt{3}$

b) $y\text{-int} = 1$

$\therefore y = \frac{1}{\sqrt{3}}x + 1$

Exercise 5F: *page 142*
Questions: *all*